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Surreal dimensions

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Abstract

In this paper we use Conway's surreal numbers to define a refinement of the box-counting dimension of a subset of a metric space. The surreal dimension of such a subset is well-defined in many cases in which the box-counting dimension is not. Surreal dimensions refine box-counting dimensions due to the fact that the class of surreal numbers contains infinitesimal elements as well as every real number. We compute the surreal dimensions of generalized Cantor sets, and we state some open problems.

Keywords: Box-counting dimension; Surreal numbers

1. Introduction

Let S be a non-empty subset of a metric space. Suppose that for each real $\rho > 0$, there is a minimal finite number $N(\rho)$ of closed balls of radius ρ needed to cover S. This will be the case, for example, if the closure of S is compact. The *box-counting dimension* of S compares the rate of growth of the function

$$x \to N_1(x) = N(1/x)$$

as $x \to +\infty$ to that of the function

$$x \to x^d$$

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when d is a positive real number. Define

$$d_-(S) = \liminf_{x\to\infty} \frac{\log(N_1(x))}{\log(x)} \quad \text{and} \quad d_+(S) = \limsup_{x\to\infty} \frac{\log(N_1(x))}{\log(x)}, \tag{1.1}$$

where these limits may be $+\infty$. Following [3, Chapter 3], we will call $d_-(S)$ (respectively $d_+(S)$) the lower (respectively upper) box-counting dimension of S. If $d_-(S) = d_+(S)$, then their common value $d(S)$ is called the box-counting dimension of S; otherwise $d(S)$ is not well-defined.

The object of this paper is to generalize the box-counting dimension by associating to S a *surreal number* $d_\lambda(S)$. We will see that $d_\lambda(S)$ is well defined in many cases in which $d(S)$ is not, i.e., in which $d_-(S) < d_+(S)$. When both $d_\lambda(S)$ and $d(S)$ are well-defined, $d_\lambda(S)$ will in general measure more precisely than $d(S)$ the growth of $N_1(x)$ as $x \to +\infty$.

We recall in Section 2 Conway's definition of the class **No** of surreal numbers. (The name "surreal numbers" was coined Knuth [5].) The surreal numbers form a field, i.e., a field whose domain is a proper Class (cf. [1, p. 4]). Members of **No** include the real numbers and a vast collection of elements which are *infinitesimal* or *infinitely large* in comparison to every positive real number. (An element a of **No** is infinitesimal with respect to another element b if $0 < na < b$ for all positive integers n; one then says b is infinitely large in comparison to a.)

We will use surreal numbers to label the elements of well ordered sets of test functions. There are two degrees of freedom involved in doing this. One can choose different well ordered sets of test functions, and one can label these using different collections of surreal numbers. We call a labeling λ of this kind a *yardstick*; a formal definition is given in Definition 2.3 of Section 2.

To use a yardstick λ to measure the rate of growth of the function $x \to N_1(x)$, one lets $\mathcal{L}$ (respectively $\mathcal{R}$) be the set of test functions used in the definition of λ which are asymptotically less than (respectively greater than) $x \to N(1/x)$. Since surreal numbers are defined as generalized Dedekind cuts, there is a unique surreal number $d_\lambda(S)$ defined by the sets of surreal numbers $\lambda(\mathcal{L})$ and $\lambda(\mathcal{R})$. This $d_\lambda(S)$ is the surreal dimension of S with respect to λ; a formal definition is given in Definition 2.4 of Section 2.

One can think of $d_\lambda(S)$ as the 'simplest' compromise, in the sense of [1, Theorem 2.11], between the lower and upper bounds on the dimension of S provided by $\lambda(\mathcal{L})$ and $\lambda(\mathcal{R})$. Because surreal numbers make such a compromise by virtue of their definition, $d_\lambda(S)$ is well-defined even when $d_-(S) < d_+(S)$, i.e., even when $d(S)$ is not defined. This approach to defining dimensions can be viewed as a form of Occam's razor, since it consists of choosing the 'simplest' (surreal) number consistent with data one has concerning S. In Example 4.1 we construct examples of S for which $d_-(S) < d_+(S)$.

To illustrate how surreal numbers can be used to measure growth rates, consider the function $x \to x^n$ of positive real x when n is a fixed non-negative integer. This is the similarity ratio of n-dimensional Euclidean volume. The number most readily associated to the growth rate of this function is the integer n. It is less obvious which (surreal) numbers should be associated to the growth rates of functions such as $x \to \mathrm{e}^x$. Since n is the exponent appearing in $x \to x^n$, a naive approach would be to look for a surreal number α such that $x^\alpha = \mathrm{e}^x$ for all sufficiently large real x. Unfortunately, no such α exists, so one must simply choose a plausible α to associate to the rate of growth of $x \to \mathrm{e}^x$. Clearly this α should be greater than every integer. In [2, p. 299], Conway and Guy propose using for α the 'simplest' surreal number ω greater than every integer. The laws of exponents now suggest that since $x = x^1 = \log(\exp(x))$, we should let the growth rate of $x \to \log(x)$ correspond to $1/\omega$, since formally $(x^\omega)^{1/\omega} = x$. Similarly, the function $x \to \log(\log(x))$ should correspond to $1/\omega^2$, since formally one has $(x^{1/\omega})^{1/\omega} = x^{1/\omega^2}$. These choices lead to the *logarithmic yardsticks* discussed in Section 3. The nth logarithmic yardstick associates to the growth rate of

$$x \to x^{a_0} \cdot \bigl(\log(x)\bigr)^{a_1} \cdots \bigl(\log^{(n)}(x)\bigr)^{a_n}$$

as $x \to +\infty$ the surreal number

$$a_0 + a_1/\omega + \cdots + a_n/\omega^n$$

if $a_0, \ldots, a_n$ are real numbers and the smallest index i for which $a_i \neq 0$ has the property that $a_i > 0$.

The logarithmic yardstick λ_0 defined by setting $n = 0$ in the above discussion involves only the test functions $x \to x^{a_0}$ when a_0 is a positive real number. It yields the ordinary dimension when $d_-(S) = d_+(S)$, but will give a surreal number which is a dyadic rational number when $d_-(S) < d_+(S)$. Indeed, we show in Theorem 3.3 the particular compromise $d_{\lambda_0}(S)$ makes between the classical box-counting dimensions $d_-(S)$ and $d_+(S)$. The nth yardstick for $n > 0$ allows a refinement of dimension with respect to a wider asymptotic class, in addition to resolving cases in which lower dimension is unequal to upper dimension.

In Section 4 we show that the surreal dimension of the union of two sets may be strictly larger than the surreal dimension of either set. In Section 5 we determine the surreal dimension of various generalized Cantor sets with respect to the logarithmic yardsticks defined in Section 3. Generalized Cantor sets arise in harmonic analysis in the study of sets of uniqueness [4,6], and it would be interesting to investigate the relevance of surreal dimensions to the study of sets of uniqueness.

We should emphasize, though, that there are many other possible yardsticks than the logarithmic ones studied in Sections 3–5. Instead of using iterates of $\log(x)$, for example, one could have used iterates of some other increasing function of x which tends to $+\infty$ as $x \to +\infty$ more slowly than any positive power of x. More generally, one could have used several functions and their various compositions, provided the growth rates of these functions with x are well-ordered in a suitable sense. One also has a great deal of freedom in choosing which surreal numbers to associate to each test function. For example, in discussing logarithmic yardsticks, $1/\omega$ could have been replaced by any positive surreal number less than every positive real number.

The problem of how to choose natural yardsticks arises whenever one would like to assign a numerical size to the growth rate of a function. This problem is related to the question of how to assign a precise meaning to sums of asymptotic series, which has recently been studied using surreal numbers by Kruskal [1, pp. 225–228]. In Section 6 we discuss one specific question having to do with yardsticks constructed from the functions $x \to x^r$ for real r and $x \to \log(x)$. There is a large literature concerning the box-counting dimensions of various sets S (cf. [3] and its references), and it seems natural now to consider the surreal dimensions of these S with respect to various yardsticks.

2. Surreal numbers and dimensions

Recall that a surreal number $\alpha = \{L \mid R\}$ and an ordering $\leqslant$ on surreal numbers may be defined inductively in the following way [1]. Let L and R be (possibly empty) collections of surreal numbers. Suppose that for each $\alpha^L \in L$ and $\alpha^R \in R$ one has $\alpha^R \nleqslant \alpha^L$, meaning that it is not the case that $\alpha^R \leqslant \alpha^L$. Then the pair $\alpha = \{L \mid R\}$ is defined by induction to be a surreal number. One defines $\alpha \leqslant \beta = \{L' \mid R'\}$ if and only if no $\beta^R \in R'$ satisfies $\beta^R \leqslant \alpha$ and no $\alpha^L \in L$ satisfies $\beta \leqslant \alpha^L$. One says α is identical to β if $L = L'$ and $R = R'$; if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$ then α is said to be equal to β. For further details, see [1]. In particular, one can define the addition and multiplication of surreal numbers in such a way that they form totally ordered field containing the real numbers.

Definition 2.1. Let $\mathbb{R}_{>c}$ be the set of real numbers greater than the real constant c. Define $\mathcal{M}_c$ to be the set of all non-decreasing functions $f : \mathbb{R}_{>c} \to \mathbb{R}_{>0}$, and let $\mathcal{M}$ be the union of $\mathcal{M}_c$ over all $c > 0$. Define $\mathcal{C}$ to be the subset of $f \in \mathcal{M}$ which are continuous, monotonically increasing and have $\lim_{x\to\infty} f(x) = +\infty$. Define an equivalence relation $\equiv$ on $\mathcal{M}$ by saying $f \equiv g$ if $f(x) = g(x)$ for all sufficiently large real x. Let $[f]$ be the equivalence class of f. Define $\mathcal{E} = \{[f]\colon\ f \in \mathcal{M}\}$ and $\mathcal{E}_{\mathcal{C}} = \{[f]\colon\ f \in \mathcal{C}\}$. Say $[f] \geqslant [g]$ if $f(x) \geqslant g(x)$ for all sufficiently large x.

If $f \in \mathcal{C}$, the inverse function $g(y)$ to $f(x)$ is well-defined and continuous for all sufficiently large y. The following Lemma is now clear.

Lemma 2.2. *Composition of functions gives the set $\mathcal{E}$ (respectively $\mathcal{E}_{\mathcal{C}}$) the structure of a partially ordered semigroup (respectively group), with identity element $[f_1]$ when $f_1(x) = x$ for all $x > 0$.*

Definition 2.3. A yardstick is an order preserving bijection $\lambda : T \to \lambda(T)$ between a totally ordered subset T of $\mathcal{E}$ and a subset $\lambda(T)$ of the class of surreal numbers.

Suppose now that S is a non-empty subset of a metric space, and that for each $\rho > 0$ there is a finite minimal number $N(\rho)$ of closed balls of radius $\rho > 0$ needed to cover S. Define $N_1(x) = N(1/x)$ for all $x > 0$. We can now define the surreal dimension $d_\lambda(S)$ of S with respect to the choice of yardstick $\lambda : T \to \lambda(T)$.

Definition 2.4. Let $T_-(S)$ (respectively $T_+(S)$) be the set of all $[f(x)] \in T$ such that $\lim_{x\to+\infty} f(x)/N_1(x) = 0$ (respectively $\lim_{x\to+\infty} N_1(x)/f(x) = 0$). The surreal dimension $d_\lambda(S)$ of S relative to λ is the surreal number

$$\bigl\{\lambda\bigl(T_-(S)\bigr) \bigm| \lambda\bigl(T_+(S)\bigr)\bigr\}.$$

We now prove some simple properties of the surreal dimension. For a discussion the corresponding properties for other notions of dimension, see [3, p. 37].

Theorem 2.5. *Suppose S and S' are non-empty subsets of a fixed metric space M, and that for each $x > 0$ there is a minimal finite number $N_{1,S}(x)$ (respectively $N_{1,S'}(x)$) of closed balls of radius $1/x$ needed to cover S (respectively S'). Let $\lambda : T \to \lambda(T)$ be a fixed yardstick such that*

$$\lim_{x\to\infty} f(x) = +\infty \quad \textit{for each } \bigl[f(x)\bigr] \in T. \tag{2.1}$$

(a) (Normalization) *If S is finite, then $d_\lambda(S) = \{\emptyset \mid \lambda(T)\}$ is the simplest surreal number less than each element of $\lambda(T)$.*
(b) (Monotonicity) *If $S' \subset S$ then $d_\lambda(S') \leqslant d_\lambda(S)$.*
(c) (Weak Stability) $\max(d_\lambda(S), d_\lambda(S')) \leqslant d_\lambda(S \cup S')$, *with equality if one of S or S' is a finite set.*
(d) (Lipschitz invariance) *Suppose $M = \mathbb{R}^n$ with the Euclidean metric, and that $h : M \to M$ is a bi-Lipschitz map. Then $d_\lambda(S) = d_\lambda(h(S))$. In particular, this is the case if h is a translation, a rotation or an affine transformation.*
(e) (Open sets) *Suppose M is a smooth real manifold of dimension n, that S is a non-empty open subset of M. Then*

$$T_-(S) = \Bigl\{\bigl[f(x)\bigr] \in T\colon \lim_{x\to\infty} f(x)/x^n = 0\Bigr\} \quad \textit{and}$$
$$T_+(S) = \Bigl\{\bigl[f(x)\bigr] \in T\colon \lim_{x\to\infty} x^n/f(x) = 0\Bigr\}.$$

Proof. If S is finite, then (2.1) implies $T_-(S) = \emptyset$ and $T_+(S) = T$, which gives (a). Property (b) is clear from the fact that $N_{1,S'}(x) \leqslant N_{1,S}(x)$ for all x if $S' \subset S$. The inequality $\max(d_\lambda(S), d_\lambda(S')) \leqslant d_\lambda(S \cup S')$ in part (c) follows from part (b). Suppose S is a finite set. To prove $\max(d_\lambda(S), d_\lambda(S')) = d_\lambda(S \cup S')$, part (b) implies it will suffice to show

$$d_\lambda(S') = d_\lambda(S \cup S'). \tag{2.2}$$

Since S is finite, we have

$$N_{1,S'}(x) \leqslant N_{1,S\cup S'}(x) \leqslant N_{1,S'}(x) + |S|. \tag{2.3}$$

Assumption (2.1) together with (2.3) imply $T_-(S') = T_-(S \cup S')$ and $T_+(S') = T_+(S \cup S')$, from which (2.2) is clear. To prove part (d), observe that since h is bi-Lipschitz, there is a positive constant c independent of $x > 0$ such that each closed ball B of radius $1/x$ in $M = \mathbb{R}^n$ has the property that $h(B)$ and $h^{-1}(B)$ can be covered by at most c closed balls of radius $1/x$. It follows that

$$N_{1,S}(x) \leqslant cN_{1,h(S)}(x) \quad \text{and} \quad N_{1,h(S)}(x) \leqslant cN_{1,S}(x).$$

Hence (2.1) implies $T_-(S) = T_-(h(S))$ and $T_+(S) = T_+(h(S))$, so (d) holds. Part (e) is clear from the fact that under the hypotheses in (e), there is a positive real constant c such that $x^n/c \leqslant N_1(x) \leqslant cx^n$. □

Remark 2.6. In Proposition 4.4, we show that equality in (c) need not hold for arbitrary S and S'. In defining $d_\lambda(S)$, one could proceed slightly differently by letting $\widetilde{T}_-(S)$ (respectively $\widetilde{T}_+(S)$) be the set of all $[f(x)] \in T$ such that $f(x) < N_1(x)$ (respectively $N_1(x) < f(x)$) for all sufficiently large x. Let $\tilde{d}_\lambda(S)$ be the surreal number $\{\lambda(\widetilde{T}_-(S)) \mid \lambda(\widetilde{T}_+(S))\}$. The constant $\tilde{d}_\lambda(S)$ is less stable in S than $d_\lambda(S)$, in the following sense. Even if one of S or S' are finite, one need not have $\max(\tilde{d}_\lambda(S), \tilde{d}_\lambda(S')) = \tilde{d}_\lambda(S \cup S')$. This is because the inequalities in (2.3) are not sufficient in general to imply $\widetilde{T}_-(S') = \widetilde{T}_-(S \cup S')$ and $\widetilde{T}_+(S') = \widetilde{T}_+(S \cup S')$. Similarly, one cannot conclude under the hypotheses of part (d) of Theorem 2.5 that $\tilde{d}_\lambda(S) = \tilde{d}_\lambda(h(S))$.

3. Logarithmic yardsticks

In this section we define yardsticks using products of real powers of x and of finite iterations of $\log(x)$, and we discuss how to compute the associated surreal dimensions.

For any integer $n \geqslant 0$ let Ω_n be the set of positive surreal numbers of the form

$$a = \sum_{i=0}^{n} a_i \omega^{-i} \tag{3.1}$$

in which $a_i \in \mathbb{R}$ and $\omega = \{0, 1, 2, \ldots\}$. By [1, p. 12],

$$\omega^{-1} = \{0 | 1, 1/2, 1/4, 1/8, \ldots\}$$

is the 'simplest' number between 0 and every positive real number, in the sense of [1, Theorem 2.11]. Let $\Omega_\infty = \bigcup_{n=0}^{\infty} \Omega_n$.

Let $\log^{(0)}(x) = x$ for all $x > 0$. By induction on integers $i \geqslant 0$, define a function $\log^{(i)}(x)$ for sufficiently large x by $\log^{(i+1)}(x) = \log(\log^{(i)}(x))$. Since the surreal number ω is transcendental over $\mathbb{R}$, we have an injective map $\pi_n : \Omega \to \mathcal{E}_C$ defined by $\pi_n(a) = [\pi_n(a)(x)]$, where $\pi_n(a)(x)$ is the function defined for sufficiently large real x by

$$\pi_n(a)(x) = \prod_{i=0}^{n} \bigl(\log^{(i)}(x)\bigr)^{a_i}. \tag{3.2}$$

Let $\pi_\infty : \Omega_\infty \to \mathcal{E}_{\mathcal{C}}$ be the function defined by the π_n for all n. Let T_n be the image of π_n for all $n \leqslant \infty$. Since π_n defines an order preserving bijection between Ω_n and T_n, we can make the following definition:

Definition 3.1. Suppose $n \geqslant 0$ is an integer or that $n = \infty$. The nth logarithmic yardstick is the set theoretic inverse function $\lambda_n : T_n \to \Omega_n$ to $\pi_n : \Omega_n \to T_n$.

Thus T_n for finite n consists of all equivalence classes of finite products (3.2) of real powers of iterated logarithm functions, in which some $a_i \neq 0$, and the smallest index i for which $a_i \neq 0$ has $a_i > 0$. The nth logarithmic yardstick λ_n associates to the equivalence class $[\pi_n(a)(x)]$ of the function $\pi_n(a)(x)$ in (3.2) the surreal number

$$a = a_0 + a_1/\omega + \cdots + a_n/\omega^n.$$

For $n = \infty$ we define $\lambda_\infty : T_\infty \to \Omega_\infty$ to be take the union of all the λ_j for positive integers j.

When $n = 0$, T_0 is simply $\{[x^r]\colon\ 0 < r \in \mathbb{R}\}$, and the yardstick λ_0 sends $[x^r]$ to the real number r. The surreal dimension $d_{\lambda_0}(S)$ of a set S compares to the classical box-counting dimensions $d_\pm(S)$ of (1.1) in the following way.

Definition 3.2. Suppose the classical upper dimension $d_+(S)$ defined just after (1.1) is finite. Let $I(S)$ be the union of the (possibly empty) open real interval $(d_-(S), d_+(S))$ with one, both or none of the sets $\{d_-(S)\}$ and $\{d_+(S)\}$, according to the following rule. One should not include $d_-(S)$ (respectively $d_+(S)$) in $I(S)$ if and only if $\lim_{x\to\infty} x^{d_-(S)}/N_1(x) = 0$ (respectively $\lim_{x\to\infty} N_1(x)/x^{d_+(S)} = 0$).

Theorem 3.3. *If the classical box-counting dimension $d(S)$ exists, then $d_{\lambda_0}(S) = d(S)$. Suppose now that $d(S)$ does not exist and $d_+(S)$ is finite. Then $d_{\lambda_0}(S)$ equals the minimal integer in $I(S)$, if there is one, and otherwise it equals the unique rational number in $\mathbb{Z}[1/2] \cap I(S)$ having minimal denominator.*

Proof. By Definition 2.4, $d_{\lambda_0}(S) = \{L \mid R\}$ when L (respectively R) is the set of real $r > 0$ for which $\lim_{x\to\infty} x^r/N_1(x) = 0$ (respectively $\lim_{x\to\infty} N_1(x)/x^r = 0$). By the definition of $d_-(S)$, one has

$$d_-(S) = \sup\bigl\{r(c)\colon\ c > 0\bigr\}$$

where

$$r(c) = \inf\{\log(N_1(x))/\log(x):\ c \leqslant x \in \mathbb{R}\}.$$

Thus $0 < r < r(c)$ implies $r \in L$, while $r \in L$ implies $r \leqslant r(c)$ for some $c > 0$. It now follows from the definition of $I(S)$ in Theorem 3.3 that L is the set of positive real numbers which are less than every element of $I(S)$. Similarly, R is the set of positive real r greater than every element of $I(S)$. Theorem 3.3 now follows from this and the fact that the recipe in the theorem describes the 'simplest' surreal number, in the sense of [1, Theorem 2.11], which is greater than every element of L and less than every element of R. This recipe is well-known; one can prove it using [1, Theorems 2.11, 2.12, 2.13]. □

Remark 3.4. The surreal dimension $d_{\lambda_0}(S)$ represents a compromise between $d_-(S)$ and $d_+(S)$ which is different, in general, from the average of $d_-(S)$ and $d_+(S)$. One heuristic reason for preferring $d_{\lambda_0}(S)$ is that if $d_-(S) < d_+(S)$, then these numbers alone do not provide enough information to justify taking any particular weighted average of $d_-(S)$ and $d_+(S)$. Instead, by Occam's razor, one should take the 'simplest' number consistent with the range determined by $d_-(S)$ and $d_+(S)$ (or more precisely, by the interval $I(S)$ appearing in Theorem 3.3). It would be interesting to have a probabilistic rationale for the choice of 'simplest' dimension resulting from Conway's theory, e.g. one based on some notion of randomly chosen S having given values for $d_\pm(S)$.

One can extend the analysis of $d_{\lambda_0}(S)$ in Theorem 3.3 to one of $d_{\lambda_n}(S)$ for arbitrary $n \leqslant \infty$. The first step is to define the appropriate generalization $I_n(S)$ of $I(S)$. One then must compute the simplest real number which is greater (respectively less) than all surreal numbers in Ω_n which are less than (respectively greater than) every element of $I_n(S)$. We will need only the following special case in Section 5.

Hypothesis 3.5. *Suppose n is either a non-negative integer or $n = \infty$. Let $\{b_i\}_{i=0}^n$ be a sequence of real numbers with the following properties.*

(i) *The surreal number $b = \sum_{i=0}^n b_i \omega^{-i}$ is non-negative.*

(ii) *For all integers j such that $0 \leqslant j \leqslant n$ and all ordered sets of real numbers $\{a_i\}_{i=0}^j$ the following is true. Write $a = \sum_{i=0}^j a_i \omega^{-i}$. If $0 < a < b$ as surreal numbers then*

$$\lim_{x\to\infty} \frac{\pi_j(a)(x)}{N_1(x)} = 0 \tag{3.3}$$

where $\pi_j(a)(x) = \prod_{i=0}^j \log^{(i)}(x)^{a_i}$. If $b < a$ as surreal numbers, then

$$\lim_{x\to\infty} \frac{N_1(x)}{\pi_j(a)(x)} = 0. \tag{3.4}$$

Theorem 3.6. *Hypothesis* 3.5 *implies* $d_{\lambda_n}(S) = b$ *and that the box-counting dimension* $d(S)$ *of* S *exists and equals* b_0.

4. First examples

In this section we present two examples to illustrate how Theorem 3.3 can be applied. The first example shows how the interval $I(S)$ appearing in Theorem 3.3 can be a point, an open interval, a closed interval or a half-open interval. The second example shows how the surreal dimension of the union of two sets may be strictly larger than the surreal dimension of either set.

Example 4.1. Let $a = \{a_i\}_{i=1}^{\infty}$ be a sequence of 0's and 1's. For $m \geqslant 1$, let $S_m = S_m(a) \subseteq [0,1]$ consist of the real numbers $b = \sum_{i=1}^{\infty} b_i 2^{-i}$ whose base 2 digits $b_i \in \{0,1\}$ satisfy

$$b_i = 0 \quad \text{if} \quad 1 \leqslant i \leqslant m \quad \text{and} \quad a_i = 0.$$

Thus S_m is the union of $2^{t(m)}$ distinct closed intervals of the form $[c/2^m, (c+1)/2^m]$ with $c \in \mathbb{Z}$ and $0 \leqslant c < 2^m$, where $t(m)$ is the number of elements of $\{a_i\}_{i=1}^{m}$ equal to 1. Each of these intervals contains an element of the closed non-empty set

$$S = S(a) = \bigcap_{i=1}^{\infty} S_m.$$

If $2^m \leqslant x < 2^{m+1}$, then a closed interval of length $1/x$ can intersect at most 2 distinct intervals of the form $[c/2^m, (c+1)/2^m]$. This leads to the inequalities

$$2^{t(m)-1} \leqslant N_1(x) = N(1/x) \leqslant 2^{t(m+1)} \leqslant 2^{t(m)+1} \tag{4.1}$$

where, as before, $N(1/x)$ is the number of closed intervals of length $1/x$ needed to cover S. Hence

$$\frac{t(m)-1}{m+1} \leqslant \frac{\log(N_1(x))}{\log(x)} \leqslant \frac{t(m)+1}{m}. \tag{4.2}$$

Letting x and m tend to ∞, we see from (1.1) and (4.2) that

$$d_-(S) = \liminf_{m\to\infty} \frac{t(m)}{m} \quad \text{and} \quad d_+(S) = \limsup_{m\to\infty} \frac{t(m)}{m}. \tag{4.3}$$

Since $t(m) = \sum_{i=1}^{m} a_i$, we can choose the original sequence $a = \{a_i\}_{i=1}^{\infty}$ of 0's and 1's in such a way that $m \to t(m)$ is any prescribed integer valued function for which $t(1) \in \{0,1\}$ and $t(m) \leqslant t(m+1) \leqslant t(m)+1$ for all $m \geqslant 1$. This implies the following result:

Proposition 4.2. *Suppose α and β are any real numbers such that $0 \leqslant \alpha \leqslant \beta \leqslant 1$. One can choose $a = \{a_i\}_{i=1}^{\infty}$ so that*

$$d_-(S) = \liminf_{m\to\infty} \frac{t(m)}{m} = \alpha \quad \textit{and} \quad d_+(S) = \limsup_{m\to\infty} \frac{t(m)}{m} = \beta. \tag{4.4}$$

We now observe that Theorem 3.3 determines $d_{\lambda_0}(S)$ from $d_-(S) = \alpha$ and $d_+(S) = \beta$ unless $\alpha \neq \beta$ and one of α or β is a dyadic rational number of the form $l/2^h$. In the latter case, one must decide whether α or β lie in the interval $I(S)$ defined in Theorem 3.3, and this depends on finer information concerning the sequence $\{a_i\}_{i=1}^{\infty}$. Suppose, for example, that $\alpha \neq \beta$ and that $\alpha = l/2^h$. Then $\alpha \notin I(S)$ if and only

$$\lim_{x\to\infty} x^{\alpha}/N_1(x) = 0. \tag{4.5}$$

Choosing m so that $2^m \leqslant x < 2^{m+1}$ as before, we see from (4.1) that

$$\frac{2^{m\alpha}}{2^{t(m)+1}} \leqslant \frac{x^{\alpha}}{N_1(x)} \leqslant \frac{2^{(m+1)\alpha}}{2^{t(m)-1}}. \tag{4.6}$$

The upper and lower bounds in (4.6) go to 0 as $x \to \infty$ if and only if $m\alpha - t(m) \to -\infty$ as $m \to \infty$. We have thus shown the following result concerning $I(S)$.

Proposition 4.3. *Suppose $\alpha < \beta$. If α is a dyadic rational number then*

$$\alpha \notin I(S) \quad \textit{if and only if} \quad \lim_{m\to\infty} \bigl(m\alpha - t(m)\bigr) = -\infty. \tag{4.7}$$

Similarly, if $\alpha < \beta$ and β is a dyadic rational number, then

$$\beta \notin I(S) \quad \textit{if and only if} \quad \lim_{m\to\infty} \bigl(m\beta - t(m)\bigr) = +\infty. \tag{4.8}$$

One can use (4.7) and (4.8) to construct examples in which $I(S)$ is any one of the four possible intervals (α, β), $[\alpha, \beta)$, $[\alpha, \beta]$, and $(\alpha, \beta]$.

We now show that equality in part (b) of Theorem 2.5 need not hold for arbitrary S and S', using the fact that the classical upper and lower box-counting dimensions need not behave well under unions.

Proposition 4.4. *There are subsets S and S' of the real line such that*

$$d_{\lambda_0}(S \cup S') > \max\bigl(d_{\lambda_0}(S), d_{\lambda_0}(S')\bigr).$$

Proof. We can choose sequences $a = \{a_i\}_{i=1}^{\infty}$ and $\hat{a} = \{\hat{a}_i\}_{i=1}^{\infty}$ of 0's and 1's with the following property. When $S = S(a)$ and $\widehat{S} = S(\hat{a})$ are as in Example 4.1, one has $I(S) = \{2/3\}$ while $I(\widehat{S})$ is the open interval $(1/5, 4/5)$. Thus

$$\begin{aligned} d_-(S) &= \liminf_{x\to\infty} \frac{\log(N_{1,S}(x))}{\log(x)} = 2/3 = d_+(S) \\ &= \limsup_{x\to\infty} \frac{\log(N_{1,S}(x))}{\log(x)} \end{aligned} \tag{4.9}$$

while

$$\begin{aligned} d_-\big(\widehat{S}\big) &= \liminf_{x\to\infty} \frac{\log(N_{1,\widehat{S}}(x))}{\log(x)} = 1/5 \quad \text{and} \\ d_+\big(\widehat{S}\big) &= \limsup_{x\to\infty} \frac{\log(N_{1,\widehat{S}}(x))}{\log(x)} = 4/5. \end{aligned} \tag{4.10}$$

Let

$$S' = 2 + \widehat{S} = \big\{2 + s\colon\ s \in \widehat{S}\big\}$$

so that S' is a subset of the interval $[2, 3]$. From the calculations in Example 4.1 and Theorem 2.5(d), we have

$$d_{\lambda_0}(S) = 2/3 \quad \text{and} \quad d_{\lambda_0}(S') = d_{\lambda_0}\big(\widehat{S}\big) = 1/2.$$

Consider $d_{\lambda_0}(S \cup S')$. For $x > 1$, the function $N_{1,S\cup S'}(x)$ equals $N_{1,S}(x) + N_{1,\widehat{S}}(x)$, since no closed ball of radius $1/x$ can intersect both S and S', and $N_{1,S'}(x) = N_{1,\widehat{S}}(x)$. It follows from this, (4.9) and (4.10) that

$$\begin{aligned} d_-(S \cup S') &= \liminf_{x\to\infty} \frac{\log(N_{1,S\cup S'}(x))}{\log(x)} \\ &= \liminf_{x\to\infty} \frac{\log(N_{1,S}(x) + N_{1,\widehat{S}}(x))}{\log(x)} \geqslant 2/3 \end{aligned} \tag{4.11}$$

and

$$d_+(S \cup S') = \limsup_{x\to\infty} \frac{\log(N_{1,S\cup S'}(x))}{\log(x)} = 4/5. \tag{4.12}$$

Theorem 3.3 now implies that $d_{\lambda_0}(S \cup S')$ is either $4/5$ or a dyadic rational number in the interval $[2/3, 4/5]$. In either case, we find

$$d_{\lambda_0}(S \cup S') > 2/3 = \max\big(d_{\lambda_0}(S), d_{\lambda_0}(S')\big) = \max(2/3, 1/2), \tag{4.13}$$

which proves the proposition. □

5. Generalized Cantor sets

In this section we compute the logarithmic surreal dimensions of some generalized Cantor sets (cf. [4, Chapter I], [6]).

Let $r_1, r_2, \ldots$ be a sequence of real numbers with $0 < r_m < 1/2$. Construct for each non-negative integer m a set E_m as follows. Let E_0 be the closed interval $[0, 1]$. Suppose by induction that $m \geqslant 0$ and that E_m is the disjoint union of 2^m closed intervals, each of length $\rho_m = r_1 \cdot r_2 \cdots r_m$. Form E_{m+1} by removing an open interval from the middle of each subinterval of E_m in such a way that all of the resulting closed intervals have length $\rho_{m+1} = r_1 \cdot r_2 \cdots r_{m+1}$. The set

$$E_\infty = E_m(r_1, r_2, \ldots) = \bigcap_{m=0}^{\infty} E_m$$

is a generalized Cantor set associated to $\{r_i\}_{i=1}^{\infty}$. When $r = r_1 = r_2 = \cdots$, then one simply writes $E_\infty = E_\infty(r)$. These sets are also called *symmetric perfect sets* and have relevance to harmonic analysis and sets of uniqueness, see [4] and [6].

Since E_∞ contains the endpoints of each of the 2^m closed intervals forming E_m, and these intervals have length ρ_m, we see that the minimal number $N(\rho_m)$ of closed balls of radius ρ_m needed to cover E_∞ is exactly

$$N(\rho_m) = 2^m. \tag{5.1}$$

We now consider the surreal dimension of E_∞ with respect to the yardstick $\lambda_n : T_n \to \Omega_n$ of Definition 3.1, where n is either a non-negative integer or $n = \infty$.

Proposition 5.1. *Let* $S = E_\infty$ *in Theorem* 3.6. *We can replace condition* (3.3) *by the requirement that if* $0 < a < b$ *then*

$$\lim_{m\to\infty} \frac{\pi_j(a)(1/\rho_m)}{2^m} = 0. \tag{5.2}$$

We can replace condition (3.4) *by the requirement that if* $b < a$ *then*

$$\lim_{m\to\infty} \frac{2^m}{\pi_j(a)(1/\rho_m)} = 0. \tag{5.3}$$

Proof. As before, let $N_1(x) = N(1/x)$, and let a, j and b be as in Theorem 3.6. Suppose $0 < a < b$ and that (5.2) is true. We wish to show that condition (3.3) holds. Choose a surreal number a' of the form $\sum_{i=0}^{j} a'_i \omega^{-i}$ in which the a'_i are real and $a < a' < b$. Since $1/\rho_m > 2^m$ we have $1/\rho_m \to \infty$ as $m \to \infty$. Hence for each $x \gg 0$ we can find an m such that

$$\frac{1}{\rho_m} \leqslant x < \frac{1}{\rho_{m+1}}. \tag{5.4}$$

Since (5.2) holds when a is replaced by a', for all sufficiently large x we will have

$$\pi_j(a')\left(\frac{1}{\rho_{m+1}}\right) < 2^{m+1} \tag{5.5}$$

when m is chosen as in (5.4). For large x we now deduce from (5.5) and (5.1) that

$$\pi_j(a')(x) \leqslant \pi_j(a')\left(\frac{1}{\rho_{m+1}}\right) < 2^{m+1} = 2N_1\left(\frac{1}{\rho_m}\right) \leqslant 2N_1(x). \tag{5.6}$$

Since $a < a'$, we have

$$\lim_{x\to\infty} \frac{\pi_j(a)(x)}{\pi_j(a')(x)} = 0. \tag{5.7}$$

Thus (5.6) and (5.7) implies condition (3.3) holds. The proof that (5.3) for all a such that $b < a$ implies (3.4) is similar. □

Example 5.2. The classical Cantor set is simply $E_\infty(1/3)$. We claim that

$$d_{\lambda_n}\bigl(E_\infty(1/3)\bigr) = \log(2)/\log(3) \tag{5.8}$$

for all $n \leqslant \infty$ in this case. To show this, let $b = \log(2)/\log(3)$ in Proposition 5.1 and Theorem 3.6. For all integers $m \geqslant 1$ and $j \geqslant 0$ one has

$$\rho_m = 3^{-m} \quad \text{and} \quad \pi_j(b)\left(\frac{1}{\rho_m}\right) = 3^{mb} = 2^m.$$

Since conditions (5.2) and (5.3) hold, (5.8) now results from Proposition 5.1.

Example 5.3. Suppose $r_i = \exp(-2^i)$ for all i. Then

$$\rho_m = \prod_{i=1}^{m} \exp\bigl(-2^i\bigr) = \exp\bigl(-2^{m+1} + 2\bigr).$$

Thus for all $j \geqslant 1$,

$$\pi_j\bigl(\omega^{-1}\bigr)\left(\frac{1}{\rho_m}\right) = \log\left(\frac{1}{\rho_m}\right) = 2^{m+1} - 2.$$

Proposition 5.1 now implies $d_{\lambda_n}(E_\infty) = \omega^{-1}$ for all $1 \leqslant n \leqslant \infty$, while $d_{\lambda_0}(E_\infty) = d_-(E_\infty) = d_+(E_\infty) = 0$.

Proposition 5.4. *Suppose n is a positive integer or $n = \infty$, and that $b = \sum_{i=0}^{n} b_i\omega^{-i}$ is a surreal number such that the b_i are real and $0 \leqslant b \leqslant 1$. There is a choice of $r_1, r_2, \ldots$ such that $d_{\lambda_n}(E_\infty) = b$.*

Proof. Suppose first that $n < \infty$. If $b = 0$, we can choose the r_i to decrease sufficiently rapidly to 0 so that (5.3) holds for all $a > 0$, where $1/\rho_m = \prod_{i=1}^{m} r_i^{-1}$. Proposition 5.1 then shows $d_{\lambda_n}(E_\infty) = 0$. Suppose now that $0 < b \leqslant 1$. For sufficiently large x, the function $x \to \pi_n(b)(x)$ is a continuous, monotonically increasing function of x. For $y \geqslant 1$ define

$$f(x, y) = \frac{\pi_n(b)(yx)}{\pi_n(b)(x)} = \prod_{i=0}^{n} \frac{\log^{(i)}(yx)^{b_i}}{\log^{(i)}(x)^{b_i}} = y^{b_0} \cdot \prod_{i=1}^{n} \frac{\log^{(i)}(yx)^{b_i}}{\log^{(i)}(x)^{b_i}}.$$

If x is sufficiently large, then because $0 < b \leqslant 1$, the function of a real variable $y \geqslant 2$ defined by $y \to f(x, y)$ is continuous and increases monotonically to ∞, with $f(x, 2) \leqslant 2$. Thus for sufficiently large x,

$$(2, \infty) \subset \bigl\{f(x, y)\colon\ y > 2\bigr\}. \tag{5.9}$$

We now choose $M \geqslant 1$ and any values of $r_1, \ldots, r_M$ in $(0, 1/2)$ so that (5.9) is true provided $x \geqslant \rho_M$. For $m \geqslant M$ we have

$$\frac{1}{\rho_{m+1}} = \frac{1}{\rho_m} \cdot \frac{1}{r_{m+1}}$$

where $y = 1/r_{m+1}$ can be chosen to be any real number in $(2, \infty)$. Thus (5.9) implies that we can choose the r_m for $m \geqslant M$ so that $0 < r_m < 1/2$ and

$$f\left(\frac{1}{\rho_m}, \frac{1}{r_{m+1}}\right) = \frac{\pi_n(b)(1/\rho_{m+1})}{\pi_n(b)(1/\rho_m)} = 2^{1+2^{-m}}.$$

This equality for all $m \geqslant M$ implies that

$$c_0 < \pi_n(b)\left(\frac{1}{\rho_m}\right) \cdot 2^{-m} < c_2 \tag{5.10}$$

for some positive constants c_0 and c_1 independent of m. From (5.10) we see that conditions (5.2) and (5.3) hold, so $d_{\lambda_n}(E_\infty) = b$. If $n = \infty$, a similar argument leads the existence of a sequence $r_1, r_2, \ldots$ of the required kind, but one must use $\pi_\infty(\sum_{i=0}^{n'} b_i \omega^{-i})(x)$ for a suitable sequence of integers n' tending to ∞ because $\pi_\infty(b)(x)$ is not defined in general. $\square$

Remark 5.5. Note that Hypothesis 3.5 and Theorem 3.6 imply that $d_-(E_\infty) = d_+(E_\infty) = b_0$. In particular, this is true for the E_∞ constructed in the proof of Proposition 5.4. However, there may be other generalized Cantor sets E_∞ for which $d_{\lambda_n}(E_\infty) = b$ but for which Hypothesis 3.5 does not hold.

6. Choosing yardsticks

A basic problem in studying surreal dimensions is to choose a yardstick $\lambda : T \to \lambda(T)$ which is appropriate for a given set S. Let $\pi : \lambda(T) \to T \subset \mathcal{E}$ be the inverse function of λ, so that π takes surreal numbers to equivalence classes of functions. One can consider π which are maximal, in a suitable sense, among all choices satisfying various axioms. For example, suppose one wishes to view $\pi(\alpha)$ for a surreal number $\alpha \in \lambda(T)$ as the equivalence class of a function $x \to x^\alpha$, the latter just being a heuristic notation at this point. Then one might require the axioms

$$\pi(\alpha_1 + \alpha_2) = \pi(\alpha_1) \cdot \pi(\alpha_2) \tag{6.1}$$

and/or

$$\pi(\alpha_1 \cdot \alpha_2) = \pi(\alpha_1) \circ \pi(\alpha_2). \tag{6.2}$$

On the right in (6.1) one has what we will call the ordinary product of elements of $\mathcal{E}$, arising from the usual multiplication of functions. The binary operation on the right in (6.2) arises from composition of functions. However, since multiplication of surreal numbers is commutative, one sees that (6.2) forces

$$\pi(\alpha_1) \circ \pi(\alpha_2) = \pi(\alpha_2) \circ \pi(\alpha_1) \tag{6.3}$$

which strongly limits the possibilities for π. One could weaken (6.2) by requiring it to hold only when α_2 is in certain sets, e.g. for $\alpha_2 \in \{\alpha_1^i\}_{i=0}^{\infty}$, or for $\alpha_2 \in \{\alpha_1^i\}_{i=-\infty}^{\infty}$.

A different approach is to decide on some subset $\mathcal{W}$ of $\mathcal{E}$ which contains test functions that would be appropriate to study the dimension of a set S. One can then try to pick out a large totally ordered subset T of $\mathcal{W}$ together with a labeling of this subset via a yardstick $\lambda : T \to \lambda(T)$. Recall from Section 1 that $\mathcal{E}_{\mathcal{C}}$ is the subset of $\mathcal{E}$ represented by continuous, monotonically increasing functions $f(x)$ for which $\lim_{x\to\infty} f(x) = +\infty$. As noted in Section 1, $\mathcal{E}_{\mathcal{C}}$ becomes a group under composition of functions.

Definition 6.1. Suppose $\mathcal{W}_0$ is a subset of $\mathcal{E}_{\mathcal{C}}$. The envelope of $\mathcal{W}_0$ is the smallest subset $\mathcal{W}$ of $\mathcal{E}_{\mathcal{C}}$ which contains $\mathcal{W}_0$ and is closed under taking compositions, inverse functions and ordinary products of elements of $\mathcal{W}$.

To extend the yardsticks used in Section 3, we will ask the following question:

Question 6.2. Let $\mathcal{W}$ be the envelope in $\mathcal{E}_{\mathcal{C}}$ of the set

$$\mathcal{W}_0 = \{[x^r]\colon 0 < r \in \mathbb{R}\} \cup \{[\log(x)]\}.$$

Is $\mathcal{W}$ a totally ordered subset of $\mathcal{E}_{\mathcal{C}}$?

The set $\mathcal{W}$ defined in this question contains a great many useful test functions, e.g. $\exp(\sqrt{\log(x)})$, beyond the ones arising from the logarithmic yardsticks of Section 3. Rather than considering all of $\mathcal{W}$, it may be useful to consider those subsets of $\mathcal{W}$ which can be reached by a given number of ordinary products, compositions and inverse function operations. One then still has the problem of how to label totally ordered subsets T of $\mathcal{W}$ by yardsticks $\lambda : T \to \lambda(T)$. Such yardsticks could be useful in many problems in which one needs a numerical measure of the rate of growth of a function.

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